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SUBMERSIVE AND UNIPOTENT GROUP QUOTIENTS AMONG SCHEMES OF A COUNTABLE TYPE OVER A FIELD k

BY

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ABSTRACT. An algebraic group G is called submersive if every quotient in affine schemes c^G: Spec $A \longrightarrow$ Spec A^G which is surjective is also submersive. We prove that every unipotent group is submersive. Suppose G is submersive. We show that if c^G(Spec A) is open in Spec A^G or if some restrictions on the action of G on A are made, c^G is a topological quotient. A criterion for semisimplicity of points is extended to the case where G is unipotent. Finally, applications of the theory are provided.

0. Introduction. Let X be an affine scheme of finite type over a field k and suppose that G is an irreducible algebraic group over k which has a closed action on X via a k morphism σ: $G \times X \longrightarrow X$. If $X =$ Spec A, A^G consists of the functions in A invariant under G and $X^G =$ Spec A^G, we call the map c^G: $X \longrightarrow X^G$ induced by the inclusion $A^G \longrightarrow A$ the *algebraic group quotient of X by G*. It is easily seen that if G is affine, the mapping c^G is the coequalizer of σ and P_2, the projection of $G \times X$ onto X, in the category of affine schemes of a countable type over k (see [2]).

Definition 1. Let X be a free variable. If c^G: $X \longrightarrow X^G$ is surjective implies that c^G: $X \longrightarrow X^G$ is submersive, we say that G is *submersive*.

Every reductive group is submersive. See Mumford [6, p. 27, Theorem 1.1]. In §4, we show that every unipotent group is submersive.

The basic notation to be used here is that of Mumford [6]. However, we present now some of the conceptions to be found in Mumford [6] in a more compact form using some category theory.

Definition 2. Let D_1 be a subcategory of D_2 and D_2 be a subcategory of D_3 (thus, D_1, D_2 and D_3 are categories). Suppose that f, g: $Y_1 \rightrightarrows Y_2$

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are two maps in D_1. A D_3 *coequalizer in* D_2 *relative to* D_1 is a map $c: Y_2 \to Y_3$ in D_2 such that, for each map $h: Y_2 \to Y_4$ in D_3 satisfying $h \circ f = h \circ g$, there is a unique map $i: Y_3 \to Y_4$ in D_3 with $i \circ c = h$.

The most important theorem of this paper is the theorem that follows. In the paper $A^G \neq k$.

Theorem 1. *Let X and X^G be those affine schemes and G be that algebraic group considered above. Assume that G is submersive and affine. Then, under each of the following conditions:*

(i) *$c^G(X)$ is open in X^G,*

(ii) *for each $h \in A$ and each $\sigma \in G$, there is a sequence of functions $h_1, \dots, h_n$ in A such that*

$$h^\sigma = h + h_1, \quad h_1^\sigma = h_1 + h_2, \dots, h_{n-1}^\sigma = h_{n-1} + h_n$$

with $h_n \in A^G$, $c^G: X \to c^G(X)$ is a <u>Top</u> *and* <u>Sch</u> *coequalizer of the maps*

$$\sigma, P_2: G \times X \rightrightarrows X$$

in <u>Sch</u> *relative to* <u>Aff</u>. *Here,* <u>Top</u> *is the category of topological spaces,* <u>Sch</u> *is the category of schemes and* <u>Aff</u> *is the category of affine schemes.*

In §2 we show that condition (ii) of Theorem 1 implies condition (i). Then, in §6, we provide some examples where condition (ii) is satisfied.

Corollary 1. *Let X and X^G be those affine schemes and G be that algebraic group of Theorem 1. Suppose that G is a unipotent group. Then, $c^G: X \to c^G(X)$ is a* <u>Top</u> *and* <u>Sch</u> *coequalizer of σ and P_2 in* <u>Sch</u> *relative to* <u>Aff</u>.

Proof. In §4, we show that every unipotent group is submersive. Apply Theorem 1.

Corollary 2. *Let X and X^G be those affine schemes and G be that algebraic group of Theorem 1. Suppose that $c^G(X)$ is of finite type over k and that k is an algebraically closed field. If X is irreducible and normal and if the residue field of the generic point of $c^G(X)$ has characteristic 0, then $c^G: X \to c^G(X)$ is a geometric quotient of X by G, i.e., the mapping c^G is a* <u>Top</u> *and* <u>Sch</u> *coequalizer of σ and P_2 in* <u>Sch</u> *relative to* <u>Aff</u> *and, in addition, c^G is universally submersive.*

Proof. One applies Mumford [6, Proposition 0.2, p. 7].

Remark 1. Mumford [6, p. 6] indicates that if X is normal over k and $c^G: X \to c^G(X)$ is a geometric quotient, then $c^G(X)$ must be of finite type over k. Nagata's example (see Dieudonné [4, p. 45]) shows us that X can be of finite type and normal but that X^G may not be of finite type. A simple example illustrates, however, that $c^G(X)$ could be of finite type over k even though it is an open subscheme of an affine scheme of countable type over k. Let $i, *: k^1 \rightrightarrows k^2$ be two maps where i is a closed immersion identifying k^1 with $X = 0$, $(0, 0) \in i(k^1)$ and $*(k^1) = (0, 0)$. Then, i and $*$ have a coequalizer $k^2/k^1 = \mathrm{Spec}(k + (X))$ in $\underline{\mathrm{Sch}}$ where (X) is the ideal generated by X in $k[X, Y]$. Furthermore, $k + (X)$ is not of finite type over k; see [2]. Localize at $X \neq 0$. Then, $(k + (X))_X \simeq k[X, Y]_X$. Let $q: k^2 \to k^2/k^1$ be the quotient map. It follows that $k^2/k^1 - q(k^1) \simeq k^2 - k^1$ is of finite type over k. This suggests the following refinement of

Hilbert's Fourteenth Problem. *Is $c^G(X)$ a scheme of finite type over k?*

In order to provide some applications of Theorem 1, we will prove the next result in §5.

Theorem 2. *Let G be a unipotent affine algebraic group which acts on projective space $\mathbf{P}^n$ by a closed action over k. Consider the cone k^{n+1} lying above $\mathbf{P}^n$. If x is a point of $\mathbf{P}^n$, suppose that x' denotes a point lying above x. Note that the action of G on $\mathbf{P}^n$ extends to an action of G on k^{n+1}. Then, x is semistable if and only if the closure of the orbit of x' does not contain* 0.

For further information on the notions contained in this theorem, the reader is referred to Mumford [6, p. 50].

Examples of unipotent actions on projective spaces, which appear as linear systems, will be provided in §6.

1. A partial generalization of Chevalley's theorem.

Lemma 1. *Let $f: X \to Y$ be a dominant morphism of affine schemes defined over k and suppose that X is of finite type over k. Then, $f(X)$ contains an open subset U of Y.*

Proof. Clearly, we can assume that X and Y are reduced. Suppose that $X_1, X_2, \ldots, X_m$ are the components of X (finite in number as X is of finite type). Consider the mapping $f_i: X_i \to \overline{f(X_i)}$, the restriction of f to X_i mapped to the closure of $f(X_i)$, $i = 1, 2, \ldots, m$. If $f(X_i)$ contains an open subset U_i of $\overline{f(X_i)}$, then $f(X_i)$ also contains an affine open subset $U(g_i)$ of $\overline{f(X_i)}$, the complement of the hyperplane g_i. In that case, $f(X) =$

$\bigcup_{i=1}^{m} f(X_i)$ contains an affine open $U(g_1 g_2 \cdots g_m)$ of $\overline{f(X)} = \bigcup_{i=1}^{m} \overline{f(X_i)}$. Therefore, we can assume in Lemma 1 that X and Y are irreducible and reduced.

The remainder of the proof, modulo some minor modifications, can be found in Mumford [7, pp. 94 and 95].

Remark. When Y is of finite type, Lemma 1 follows from Chevalley's theorem which asserts (see Grothendieck [5]) that $f(X)$ is constructible.

2. $c^G(X)$ is open in X^G. We employ the notation of §0. In order to show that $c^G\colon X \to c^G(X)$ is a Top coequalizer in Sch, we must show that $c^G(X)$ is a scheme. This result is a consequence of the next lemma.

Lemma 2. *If condition* (ii) *of Theorem* 1 *holds*, $c^G(X)$ *is open in* X^G.

Proof. Lemma 1 implies that there is an affine open subset $U(g_1)$ of X^G such that $U(g_1) \subset c^G(X) \subset X^G$. If $c^G(X) = X^G$ or $c^G(X) = U(g_1)$, we are finished. Suppose, then, that $c^G(X) \neq X^G$ and $c^G(X) \neq U(g_1)$ and consider the map

$$c_1^G\colon \operatorname{Spec} A/(g_1) \to \operatorname{Spec} A^G/(g_1)$$

where (g_1) denotes the ideal generated by g_1 in A and A^G, respectively.

Suppose that c_1^G is dominant. Lemma 1 implies that there is an open subset $U(g_2)$ of Spec $A^G/(g_1)$ such that

$$U(g_2) \subset c_1^G(\operatorname{Spec} A/(g_1)) \subset \operatorname{Spec} A^G/(g_1).$$

If $(a)_1$ $U(g_2) = c_1^G(\operatorname{Spec} A/(g_1))$ or $(b)_1$ $c_1^G(\operatorname{Spec} A/(g_1)) = \operatorname{Spec} A^G/(g_1)$, we are finished. In case $(a)_1$, $c^G(X) = X^G - (V(g_1) \cap V(g_2))$ (where $V(g)$ denotes the hypersurface defined by g in X^G) and hence $c^G(X)$ is open in X^G. In case $(b)_1$, $c^G(X) = X^G$. This contradicts a previous assumption. Assume therefore that $(a)_1$ and $(b)_1$ do not hold.

If A is an integral domain, $g_1 \in A^G$ and $x = ag_1 \in A^G$, then $a \in A^G$. In this case, one sees easily that $A^G/(g_1) \to A/(g_1)$ is injective and hence c_1^G is dominant.

Claim (n). If $g_1, \ldots, g_n \in A^G$, then

$$c^G\colon \operatorname{Spec} A^G/(g_1, \ldots, g_n) \to \operatorname{Spec} A/(g_1, \ldots, g_n)$$

is injective.

Proof of Claim (n). Suppose that $g_1, g_2, \ldots, g_n \in A^G$ and

$$(\gamma) \qquad a_1g_1 + a_2g_2 + \cdots + a_ng_n = 0$$

with $a_1, a_2, \ldots, a_n \in A$. We show that there are $a_1', a_2', \ldots, a_n' \in A^G$, not all zero, such that $a_1'g_1 + a_2'g_2 + \cdots + a_n'g_n = 0$.

If $f = a_1g_1 + a_2g_2 + \cdots + a_{n-1}g_{n-1} \in A^G$, taking $a_n = -1$ and $g_n = f$ above, $f = a_1'g_1 + \cdots + a_{n-1}'g_n$ for elements $a_1', \ldots, a_{n-1}' \in A^G$. This implies that the map $A^G/(g_1, \ldots, g_{n-1}) \to A/(g_1, \ldots, g_{n-1})$ is injective and proves the claim.

Suppose that $a_1 \notin A^G$. There is then a $\sigma \in G$ such that $a_1^\sigma \neq a_1$. Condition (ii), Theorem 1, can be applied to determine infinite sequences a_j^i, $i = 0, 1, \ldots$; $j = 1, \ldots, n$, of functions such that, for $j = 1, \ldots, n$, $(a_j^i)^\sigma = a_j^i + a_j^{i+1}$ where $a_j = a_j^0$ and, for some integer $m(j)$, $a_j^{m(j)}$ is in A^G. Note that $a_j^{m(j)+1} = 0$. We can assume that a_j^i, $i < m(j)$, is not invariant under σ and that $m = \max(m(j)) = m(1)$.

Applying σ to γ, we find that

$$(a_1 + a_1^1)g_1 + (a_2 + a_2^1)g_2 + \cdots + (a_n + a_n^1)g_n = 0$$

or using γ that

$$a_1^1g_1 + a_2^1g_2 + \cdots + a_n^1g_n = 0.$$

This process can be used m times to show that

$$a_1^mg_1 + a_2^mg_2 + \cdots + a_n^mg_n = 0.$$

Since a_1^{m-1} is not invariant under σ, $a_1^m \neq 0$. Clearly, $a_1^m, a_2^m, \ldots, a_n^m \in A^G$. The claim is proven.

Continuing as we have, we obtain at stage n an affine open subset $U(g_n)$ of Spec $A^G/(g_1, g_2, \ldots, g_{n-1})$ from a dominant map c_{n-1}^G such that

$$U(g_n) \subset c_{n-1}^G(\operatorname{Spec} A/(g_1, g_2, \ldots, g_{n-1})) \subset \operatorname{Spec} A^G/(g_1, \ldots, g_{n-1}).$$

If

(a)$_n$ $U(g_n) = c_{n-1}^G(\operatorname{Spec} A/(g_1, \ldots, g_{n-1}))$, or

(b)$_n$ $c_{n-1}^G(\operatorname{Spec} A/(g_1, \ldots, g_{n-1})) = \operatorname{Spec} A^G/(g_1, \ldots, g_{n-1})$,

we are done. In case (a)$_n$, $c^G(X) = X^G - (\bigcap_{i=1}^n V(g_i))$. In case (b)$_n$ is true, we find that (b)$_{n-1}$ is true. Assume that (a)$_n$ and (b)$_n$ are not true.

Let

$$X_n = (c^G)^{-1}\left(X^G - \left(\bigcap_{i=1}^n V(g_i)\right)\right).$$

Then, X_n is an open subset of X; and because of the nature of our construction, i.e. we assume $(b)_n$ is not true, if $X \neq X_n$, then $X_n \subsetneqq X_{n+1}$. But, as X is of finite type over k and hence satisfies the ascending chain condition on open sets, there is an integer N such that $X_N = X$. It follows that

$$c^G(X) = c^G(X_N) = X^G - \left(\bigcap_{i=1}^{N} V(g_i)\right). \quad \text{Q.E.D.}$$

3. **Proof of Theorem 1.** According to Mumford [6, p. 8], a categorical quotient $(c^G(X), c^G)$ to σ and P_2 exists when the following conditions hold:

(i) $c^G \circ \sigma = c^G \circ P_2$.

(ii) $\underline{o}_{c^G(X)}$ is the sheaf of invariants of $c^G_*(\underline{o}_X)|_{c^G(X)}$.

(iii) If W is an invariant closed subset of X, $c^G(W)$ is closed in X^G; if $\{W_i\}$, $i \in I$, form a set of invariant closed subsets of X, then

$$c^G\left(\bigcap_{i\in I} W_i\right) = \bigcap_{i \in I} c^G(W_i).$$

Let $\{U_j\}_{j\in J}$ be a covering of $c^G(X)$ by affine opens. We employ the proposition below demonstrated in [3].

Proposition 1. *In the category of countably generated k algebras, localization preserves equalizer.*

As a consequence of this proposition, the maps

$$c_j^G\colon (c^G)^{-1}(U_j) \to U_j \qquad (j \in J)$$

are algebraic quotients of $(c^G)^{-1}(U_j)$ by G and surjective. Furthermore, it is not difficult to see that if (i), (ii) and (iii) hold for c_j^G and c_j^G is a topological group quotient, $j \in J$, (i), (ii) and (iii) hold for c^G and c^G is a topological group quotient. Therefore, we can assume that $c^G(X) = X^G$.

Having made the last assumption, (i) is obvious. (ii) follows from Proposition 1 and the fact that the defining property of a sheaf is an equalizing diagram. As G is submersive, (iii) will follow from the next proposition. See Mumford [6, p. 28].

Proposition 2. *If the mapping $c^G\colon X \to X^G$ is surjective and submersive and if A_i, $i \in I$, are the invariant (reduced) ideals in A corresponding to the W_i in* (iii), *then there is an equality*

$$\left(\sum_{i\in I} A_i\right)^{1/2} \cap A^G = \left(\sum_{i\in I}(A_i \cap A^G)\right)^{1/2}.$$

Proof of Proposition 2. We have

$$\left(\sum_{i\in I} A_i\right)^{1/2} \cap A^G = \left(\sum_{i\in I}\left(\bigcap_{p\in W_i} p\right)\right)^{1/2} \cap A^G = \left(\bigcap_{p\in \cap W_i} p\right) \cap A^G.$$

As c^G is submersive and surjective,

$$(*) \qquad A_i \cap A^G = \bigcap_{p\in W_i} (p \cap A^G).$$

Therefore,

$$\left(\sum_{i\in I}(a_i \cap A^G)\right)^{1/2} = \left(\sum_{i\in I}\left(\bigcap_{p\in W_i} (p \cap A^G)\right)\right)^{1/2} = \bigcap_{p\in \cap W_i} (p \cap A^G).$$

The first and last equalities imply the proposition. Note that the intersection in (*) contains all the prime ideals of A^G containing $A_i \cap A^G$. Otherwise, we cannot conclude the last equality.

To demonstrate that c^G is a topological quotient and, thus, to complete the proof of Theorem 1, we must show that under the action of G, distinct orbits are mapped by c^G into distinct points. The action of G on X is closed. Let A_1 and A_2 be the defining ideals of two distinct orbits X_1 and X_2 of X under G. Then

$$A^G = (A_1 + A_2)^{1/2} \cap A^G = ((A_1 \cap A^G) + (A_2 \cap A^G))^{1/2}.$$

Hence, $1 = f + g$, where $f, g \in A^G$, f vanishes on $c^G(X_1)$ and g vanishes on $c^G(X_2)$. It must be the case, then, that $c^G(X_1) \neq c^G(X_2)$. Q.E.D.

4. Unipotent groups are submersive. Note that every representation of a unipotent group is through a unipotent action. See Borel [1, p. 151]. The fact that unipotent groups are submersive follows, then, immediately from the next proposition.

Proposition 3. *Let G be an algebraic group such that every finite dimensional representation of G has at least one fixed point. Then G is submersive.*

Proof. Suppose that $c^G\colon X \to X^G$ is surjective where $X = \operatorname{Spec} A$ and $X^G = \operatorname{Spec} A^G$. Consider an invariant closed set W of X and let J be an invariant ideal whose zeroes equal W. J is the union of finite dimensional G invariant subspaces. See Mumford [6, p. 25]. As every finite dimensional representation of G has at least one fixed point, clearly J contains an element g_1 of A^G.

Claim. c_1^G: Spec $A/(g_1) \rightarrow$ Spec $A^G/(g_1)$ is surjective.

Proof of claim. If $q \in$ Spec $A^G/(g_1)$, q has the form $p \cap A^G$ as c^G is surjective. As q contains g_1, so does p. Therefore, $p \in$ Spec $A/(g_1)$.

Repeating this argument with c_1^G instead of c^G and $J/(g_1)$ instead of J, we obtain a surjective map c_2^G: Spec $A/(g_1, g_2) \rightarrow$ Spec $A^G/(g_1, g_2)$ where $g_2 \in J$ but $g_2 \notin (g_1)$.

Continuing with this process, at stage n, we obtain a surjective map

$$c_n^G\colon \operatorname{Spec} A/(g_1, g_2, \ldots, g_n) \rightarrow \operatorname{Spec} A^G/(g_1, g_2, \ldots, g_n)$$

where $(g_1, g_2, \ldots, g_n) \subset J$ but $g_i \notin (g_1, g_2, \ldots, g_{i-1})$. As X is of finite type over k, there is an integer N such that $J = (g_1, g_2, \ldots, g_N)$.

$$c_N^G\colon \operatorname{Spec} A/(g_1, g_2, \ldots, g_N) \rightarrow \operatorname{Spec} A^G/(g_1, g_2, \ldots, g_N)$$

is surjective. As $W =$ Spec A/J and Spec $A^G/(g_1, g_2, \ldots, g_N)$ is closed in X^G, $c^G(W)$ is a closed subset of X^G. Q.E.D.

5. **Proof of Theorem 2.** The direction: x is semistable implies that 0 is not in the closure of the orbit $O(x')$ of x' is clear. See Mumford [6, p. 50, Proposition 2.2].

Suppose that 0 is not in the closure of the orbit $O(x')$ of x'. Let W_1 be the closure of $O(x')$ and W_2 be 0. W_1 and W_2 are disjoint invariant closed subsets of K^{n+1}. To prove that x is semistable and hence to prove Theorem 2, we need to show that for some positive integer m, there is an $F \in H^0(\mathbf{P}^n, \vartheta_{\mathbf{P}^n}(m))$ such that $F(x) \neq 0$. First, we consider the affine situation where the following lemma is needed.

Lemma 3. *Suppose that the map $c^G\colon X \rightarrow X^G$ is submersive. If X_1 and X_2 are two disjoint invariant closed subschemes of X, there is an element $f \in \Gamma(X^G, \vartheta_{X^G})$ which is 1 on X_1 and 0 on X_2.*

Proof. As in the proof of Theorem 1, as c^G is submersive, applying Proposition 2, one arrives at the equation

$$A^G = (A_1 + A_2)^{1/2} \cap A^G = ((A_1 \cap A^G) + (A_2 \cap A^G))^{1/2}$$

where A_1 and A_2 are the defining ideals of X_1 and X_2, respectively. Then $1 = f + g$ where f vanishes on X_2, g vanishes on X_1 and $f, g \in A^G$. Q.E.D.

Now, as we have seen in the proof of Theorem 1, the map c^G restricts to mappings of affine opens $c_j^G: (c^G)^{-1}(U_j) \to U_j$ where $\{U_j\}_{j \in J}$ is an open affine covering of $c^G(X)$. We examine such maps in our present situation, i.e., $X = k^{n+1}$ and G is a unipotent group acting on X. Also, we place $V_j = (c^G)^{-1}(U_j)$.

Let V_j be the complement of the hypersurface g_j. As k^{n+1} is of affine type over k, only a finite number of V_j, $j = 1, \ldots, m$, are needed to cover k^{n+1}. Pick a point $Q \in W_1$. By taking an appropriate linear combination of the g_j, $j = 1, \ldots, m$, $g_0 = d_1 g_1 + d_2 g_2 + \cdots + d_m g_m$, one can guarantee that $0, Q \in V_0$ where V_0 is the complement of the hypersurface g_0. Thus, also, $W_1 \cap V_0 \neq 0$. Proposition 3 implies, as G is unipotent, that $c_0^G: V_0 = (c^G)^{-1}(U_0) \to U_0$ is submersive. By Lemma 3, there is a function f on V_0 which is one on $W_1 \cap V_0$ and such that $f(0) = 0$. f has the form $h/(g_0)^{m_j}$ where $h \in k[X_0, X_1, \ldots, X_n]$, h does not vanish on $W_1 \cap V_0$ and $h(0) = 0$. h is an invariant function as f and g_0 are invariant functions. h is, thus, constant on $O(x')$. But, then, h must be constant nonzero on W_1. Write $h = h_l + \cdots + h_{l+r}$ where h_i is homogeneous of degree i. As $h(0) = 0$, $l \geq 1$. For some i, $l \leq i \leq l + r$, h_i does not vanish at x'. Let F be that h_i. Then, F is an invariant homogeneous polynomial such that $F(x) \neq 0$. Q.E.D.

6. Examples of quotients by unipotent actions. In this section k is the field of complex numbers.

Example 1. Let U be the unipotent group which acts on k^2 and consists of matrices of the form $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. The action of U on k^2 induces an action of U on lines of the form $a_0 + a_1 X + a_2 Y = 0$ in k^2 where $a_2 \neq 0$. Under this action, lines which pass through a fixed point on the Y axis (other than $X = 0$) are identified. Thus, as it is easy to verify $X^U = c^U(X) = k^1$ where $X = k^2$ represents the collection of lines where $a_2 \neq 0$. Extending the action of U to $\mathbf{P}^2$, one sees that $X^U = \mathbf{P}^1$.

Example 2. Let H be a hyperplane in $\mathbf{P}^2$ and consider the operation of PGL(2) on $|mH|$. See Mumford [6, p. 79]. The action of PGL(2) induces an action of the unipotent subgroup U of SL(3) whose elements are of the form

$$\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

on $|mH|$. Suppose that $m = 4$. Then $|4H|$ consists of polynomials of the form

$$a_{40}X_1^4 + a_{31}X_1^3 + a_{22}X_1^2X_1^2 + a_{13}X_1X_2^3 + a_{04}X_2^4$$

$$+ a_{30}X_0X_1^3 + a_{21}X_0X_1^2X_2 + a_{12}X_0X_1X_2^2 + a_{03}X_0X_2^3$$

$$+ a_{20}X_0^2X_1^2 + a_{11}X_0^2X_1X_2 + a_{02}X_0^2X_2^2$$

$$+ a_{10}X_0^3X_1 + a_{00}X_0^4 + a_{01}X_0^3X_2.$$

The action of U on $|4H|$ sends (a_{ij}) onto

$$(a_{40}, a_{31}, a_{22}, a_{13}, a_{04}, a_{30} + ta_{31}, a_{21} + 2ta_{22}, a_{12} + 3ta_{13}, a_{03} + 4ta_{04},$$

$$a_{20} + t^2a_{22} + ta_{21}, a_{11} + 3t^2a_{13} + 2ta_{12}, a_{02} + 3ta_{03} + 6t^2a_{04},$$

$$a_{10} + t^3a_{13} + t^2a_{12} + ta_{11}, a_{01} + 4t^3a_{04} + 3t^2a_{03} + 2ta_{02},$$

$$a_{01} + t^4a_{04} + t^3a_{03} + t^2a_{02} + ta_{01}).$$

We apply Theorem 2. It follows from the above representation of the action of U that the orbit of (a_{ij}) as a point of k^{15} does not have 0 as a limit point and that condition (ii) of Theorem 1 is satisfied. Therefore, employing Theorem 1, there is a quotient of $|4H|$ by U and this quotient is topological. See [6, p. 38].

Note that the quotient of $|4H|$ by SL(3) may not exist. See Mumford [6, p. 80]. The chief problem in taking a unipotent quotient is that little is known about the interpretation of unipotent quotients.

Example 3. Suppose that we are in the same situation as in Example 2 except let U now be the full unipotent subgroup of SL(3), i.e., U consists of all matrices of the form

$$\begin{pmatrix} 1 & w & t \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}.$$

The action of U, in this case, on $|4H|$ sends (a_{ij}) into (a'_{ij}) where

$$a'_{40} = a_{40} + ua_{31} + u^2a_{22} + u^3a_{13} + u^4a_{04},$$

$$a'_{31} = a_{31} + 2ua_{22} + 3u^2a_{13} + 4u^3a_{04},$$

$$a'_{22} = a_{22} + 3ua_{13} + 6u^2a_{04},$$

$$a'_{13} = a_{13} + 4ua_{04},$$

$$a'_{04} = a_{04},$$

$$a'_{30} = a_{30} + 4wa_{40} + (t + 3wu)a_{31} + (2ut + 2wu^2)a_{22} + (3u^2t + u^3w)a_{13} + 4u^3ta_{04} + ua_{21} + u^2a_{12} + u^3a_{03},$$

$$a'_{21} = a_{21} + 3wa_{31} + (2t + 4uw)a_{22} + (6ut + 3u^2w)a_{13} + 12u^2ta_{04} + 2ua_{12} + 3u^2a_{03},$$

$$a'_{12} = a_{12} + 2wa_{22} + (3t + 3uw)a_{13} + 12uta_{04} + 3ua_{03},$$

$$a'_{03} = a_{03} + wa_{13} + 4ta_{04},$$

$$a'_{20} = a_{20} + 6w^2a_{40} + (3tw + 3w^2u)a_{31} + (4utw + t^2 + w^2u^2)a_{22} + (3ut^2 + 3u^2t)a_{13} + 6u^2t^2a_{04} + 3wa_{30} + (t + 2wu)a_{21} + (2ut + u^2w)a_{12} + 3u^2ta_{03} + ua_{11} + u^2a_{02},$$

$$a'_{11} = a_{11} + 3w^2a_{31} + (4wt + 2uw^2)a_{22} + (3t^2 + 6utw)a_{13} + 12ut^2a_{04} + 2wa_{21} + (2t + 2uw)a_{12} + 6uta_{03} + 2ua_{02},$$

$$a'_{02} = a_{02} + w^2a_{22} + 3twa_{13} + 6t^2a_{04} + wa_{12} + 3ta_{03},$$

$$a'_{10} = a_{10} + 4w^3a_{40} + (3w^2t + w^3u)a_{31} + (2wt^2 + 2utw^2)a_{22} + (t^3 + 3uwt^2)a_{13} + 4ut^3a_{04} + 3w^2a_{30} + (2wt + uw^2)a_{21} + (t^2 + 2ut)a_{12} + 3t^2ua_{03} + 2wa_{20} + (t + wu)a_{11} + 2uta_{02} + ua_{01},$$

$$a'_{01} = a_{01} + w^3a_{31} + 2w^2a_{22} + 3t^2wa_{13} + 4t^3a_{04} + w^2a_{21} + 2twa_{12} + 3t^2a_{03} + wa_{11} + 2ta_{02},$$

$$a'_{00} = a_{00} + w^4 a_{40} + w^3 t a_{31} + w^2 t^2 a_{22} + w t^3 a_{13} + t^4 a_{04} + w^3 a_{30}$$
$$+ w^2 t a_{21} + t^2 w a_{12} + t^3 a_{03} + w^2 a_{20} + w t a_{11} + t^2 a_{02} + w a_{10} + t a_{01}.$$

If the orbit of (a_{ij}) has 0 as a limit point, this implies in turn that a_{04}, a_{13}, a_{22}, a_{13}, a_{40}, a_{03}, a_{12}, a_{21}, a_{30}, a_{02}, a_{11}, a_{20}, a_{01}, a_{10}, and, finally, a_{00} are zero. Thus, again, because of Theorems 1 and 2, the quotient of $|4H|$ by U exists in schemes and this quotient is topological.

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